

GORENSTEIN CATEGORIES AND TATE COHOMOLOGY ON PROJECTIVE SCHEMES

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ABSTRACT. We study Gorenstein categories. We show that such a category has Tate cohomological functors and Avramov-Martsinkovsky exact sequences connecting the Gorenstein relative, the absolute and the Tate cohomological functors. We show that such a category has what Hovey calls an injective model structure and also a projective model structure in case the category has enough projectives.

As examples we show that if X is a locally Gorenstein projective scheme then the category $\mathcal{Q}\mathrm{co}(X)$ of quasi-coherent sheaves on X is such a category and so has these features.

1. INTRODUCTION

Tate homology and cohomology over the group ring $\mathbb{Z}G$ (with G a finite group) began with Tate's observation that the $\mathbb{Z}G$ -module \mathbb{Z} with the trivial action admits a complete projective resolution. Apparently motivated by Tate's work, Auslander showed in [5] that if A is what Bass in [7] calls a Gorenstein local ring, the finitely generated maximal Cohen-Macaulay modules can be characterized as those which admit a complete projective resolution of finitely generated projective modules. Auslander calls these modules the modules of G -dimension 0 and goes on to define the G -dimension of any finitely generated module.

In [16] an easy modification of one of Auslander's characterizations of the finitely generated modules of G -dimension 0 was given and so allows one to extend the definition to any module (finitely generated or not). Since this modified definition dualizes it seems appropriate to use the terms Gorenstein projective (corresponding to modules of G -dimension 0) and Gorenstein injective. Then there is a natural way to define Gorenstein projective and injective dimension of any module. If a module M has finite Gorenstein projective dimension n , then the n -th syzygy of M has a complete projective resolution. This complex is a homotopy invariant of M and so can be used to define Tate homological functors $\widehat{\mathrm{Ext}}_R^n(M, N)$ and $\widehat{\mathrm{Tor}}_n^R(M, N)$ for any $n \in \mathbb{Z}$. If on the other hand N has finite Gorenstein injective dimension a similar procedure can be used to define analogous functors. A. Iacob in [27] showed that if both conditions hold then the two procedures give us the same functors, i.e., that we have balance in this situation.

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In categories of sheaves there are usually not enough projectives but there are enough injectives. So we use the second approach to define Gorenstein injectives on the category of quasi-coherent sheaves on certain projective schemes. More precisely, we show that if such a scheme $X \subseteq \mathbf{P}^n(A)$ (where A is commutative noetherian) is what we call a locally Gorenstein scheme then all objects of $\mathfrak{Qco}(X)$ have finite Gorenstein injective dimension and that there is a universal bound of these dimensions. This allows us to define Tate cohomology in this situation, to get Avramov-Martsinkovsky sequences and to impose a model structure on $\mathfrak{Qco}(X)$.

The example $\mathfrak{Qco}(X)$ mentioned above is our motivation for defining Gorenstein categories. These categories will be Grothendieck categories with properties much like those of categories of modules over Gorenstein rings. But we would like our definitions to be such that nice categories of sheaves will be Gorenstein. Since categories of sheaves rarely have enough projectives we need a definition which does not involve projective objects. But such categories do have enough injectives and so the functors Ext are defined. And so projective dimensions of objects can be defined in terms of the vanishing of the Ext functors. So we define a Gorenstein category in terms of the global finitistic projective and injective dimensions of the category. After defining Gorenstein categories and proving some basic results about them, we consider examples. We show that if $X \subset \mathbf{P}^n(A)$ (where A is commutative noetherian) is a projective scheme $\mathfrak{Qco}(X)$ will be a Gorenstein category when X is a locally Gorenstein scheme.

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For all unexplained terminology see [17].

2. GORENSTEIN CATEGORIES

Our object now is to define Gorenstein categories and then exhibit some of their properties. In the section \mathcal{A} will always be a Grothendieck category with a specified generator X . The symbols Y, Z etc. will denote objects of \mathcal{A} . We will use the generator X to assign a cardinal number to every object Y . This cardinal will be $|\text{Hom}(X, Y)|$. We will now give several lemmas with the object of showing that we can do what is called set-theoretic homological algebra in a Grothendieck category.

We refer the readers to [31] for the definition and basic properties of a Grothendieck category.

Lemma 2.1. *If I is a set and if $X^{(I)} \rightarrow Y$ is an epimorphism, then there is a subset $J \subset I$ with $|J| \leq |Y|$ and such that the restriction $X^{(J)} \rightarrow Y$ is also an epimorphism.*

Proof. For such an $X^{(I)} \rightarrow Y$ we see that for each $i \in I$ we have an associated morphism $X \rightarrow Y$. Let $J \subset I$ be such that the morphisms $X \rightarrow Y$ corresponding to the $j \in J$ give all these morphisms and such that if $j \neq j'$ ($j, j' \in J$) then j and j' correspond to different morphisms. Then $|J| \leq |\text{Hom}(X, Y)| = |Y|$. Also there is a natural factorization $X^{(I)} \rightarrow X^{(J)} \rightarrow Y$. So $X^{(J)} \rightarrow Y$ is also an epimorphism. \square

Lemma 2.2. *For any Y there is an epimorphism $X^{(|Y|)} \rightarrow Y$.*

Proof. Immediate. \square

Corollary 2.3. *Given any object Y of \mathcal{A} there is a cardinal κ such that if $U \rightarrow Y$ is an epimorphism then there is a subobject $U' \subset U$ such that $U' \rightarrow Y$ is an epimorphism and such that $|U'| \leq \kappa$.*

Proof. Any such U is a quotient of $X^{(I)}$ for some set I . Then $X^{(I)} \rightarrow U \rightarrow Y$ is an epimorphism. But from the above we see that there is a subset $J \subset I$ with $|J| \leq |Y|$ such that $X^{(J)} \rightarrow U \rightarrow Y$ is an epimorphism. So let U' be the image of $X^{(J)}$ in U . So we see that it is easy to get a κ that bounds all U' that we get in this manner. \square

Corollary 2.4. *For every cardinal κ there is a set of representatives of objects Y with $|Y| \leq \kappa$.*

Proof. By having such a set we mean that we have a set of Y with $|Y| \leq \kappa$ such that every Z with $|Z| \leq \kappa$ is isomorphic to some Y in our set. Now let $|Y| \leq \kappa$. Then from the above Y is the quotient of the coproduct $X^{(\kappa)}$. But we can clearly form a set of representatives of quotients of $X^{(\kappa)}$ since \mathcal{A} is well-powered (see [30, Proposition 10.6.3]). \square

Lemma 2.5. *If $Z \subset Y$ then $|Z| \leq |Y|$.*

Proof. Immediate. \square

Lemma 2.6. *Given a cardinal κ there exists a cardinal λ such that if $|Y| \leq \kappa$ and if $Z \subset Y$ then $|Y/Z| \leq \lambda$.*

Proof. By the Corollary above there is an epimorphism $X^{(|Y|)} \rightarrow Y$ and so an epimorphism $X^{(\kappa)} \rightarrow Y$. Hence Y is isomorphic to a quotient of $X^{(\kappa)}$. Since there is a set of representatives of subobjects S of $X^{(\kappa)}$, we can take λ to be the sup of all $|X^{(\kappa)}/S|$ taken over all $S \subset X^{(\kappa)}$. \square

Lemma 2.7. *For any object Y and set I we have $|Y^{(I)}| \leq |Y^I| = |Y|^{|I|}$.*

Proof. The equality follows from the equality $\text{Hom}(X, Y^I) = \text{Hom}(X, Y)^I$. Since we are in a Grothendieck category $Y^{(I)} \rightarrow Y^I$ is a monomorphism, so we get the inequality. \square

Lemma 2.8. *For any objects Y and Z we have $|\text{Hom}(Y, Z)| \leq |Z|^{|Y|}$.*

Proof. We have an epimorphism $X^{(|Y|)} \rightarrow Y$ so $\text{Hom}(Y, Z) \subset \text{Hom}(X^{(|Y|)}, Z) = \text{Hom}(Y, Z)^{|Y|}$. But $|\text{Hom}(Y, Z)^{|Y|}| = |Z|^{|Y|}$. \square

Lemma 2.9. *If γ is an ordinal and if $(\kappa_\alpha)_{\alpha < \gamma}$ is a family of cardinal numbers, then there is a cardinal number λ such that if $(Y_\alpha)_{\alpha < \gamma}$ is a family of objects with $Y_\alpha \subset Y_{\alpha'}$ whenever $\alpha \leq \alpha', \gamma$ and such that $|Y_\alpha| \leq \kappa_\alpha$ for each $\alpha < \gamma$ then $|\bigcup Y_\alpha| \leq \lambda$.*

Proof. We have an epimorphism $X^{(|Y_\alpha|)} \rightarrow Y_\alpha$ for each α . So since we are in a Grothendieck category we have an epimorphism $X^{(\sum |Y_\alpha|)} \rightarrow \cup Y_\alpha$ and so we have an epimorphism $X^{(\sum \kappa_\alpha)} \rightarrow \cup Y_\alpha$. So $\cup Y_\alpha$ is a quotient object of $X^{(\sum \kappa_\alpha)}$. So now we appeal to Lemma 2.6 and get our λ . \square

Lemma 2.10. *Given a cardinal κ there is a cardinal λ such that if $|Y| \leq \kappa$ then $|E(Y)| \leq \lambda$ (here $E(Y)$ is an injective envelope of Y).*

Proof. We only need argue that for a given κ there is a λ such that if $|Y| \leq \kappa$ then $Y \subset E$ for an injective object E where $|E| \leq \lambda$. To show this we use a categorical version of Baer's original proof that every module can be embedded in an injective module. Since X is a generator Baer's criterion says an object E is injective if and only if it is injective for X , i.e. if for any subobject $S \subset X$ and any morphism $S \rightarrow E$ there is an extension $X \rightarrow E$. Given the object $Y = Y_0$ Baer first constructs Y_1 with $Y_0 \subset Y_1$ and such that for any $S \subset X$ and any $S \rightarrow Y_0$ there is an extension $S \rightarrow Y_1$. By his construction Y_1 is the quotient of the coproduct of Y_0 and copies of X where this is a copy of X for each $S \rightarrow Y_0$ (with $S \subset X$ arbitrary). The quotient identifies each such $S \subset X$ for a given $S \rightarrow Y_0$ with its image in Y_0 . By Lemmas 2.6 and 2.7 we see that if $\kappa = \kappa_0$ is a cardinal we can find a cardinal κ_1 such that if $|Y| = |Y_0| \leq \kappa$ then the Y_1 as constructed above is such that $|Y_1| \leq \kappa_1$. Then for any ordinal β Baer constructs a continuous chain $(Y_\alpha)_{\alpha \leq \beta}$ of objects (so $Y_\alpha \subset Y_{\alpha'}$ if $\alpha \leq \alpha' \leq \beta$ and if $\gamma \leq \beta$ is a limit ordinal then $Y_\gamma = \cup Y_\alpha$ ($\alpha < \gamma$) where Y_0 is a given Y). And then if $\alpha + 1 \leq \beta$ then we get $Y_{\alpha+1}$ for Y_α just as Y_1 is constructed from Y_0 as above and where $Y_\gamma = \cup Y_\alpha$ $\alpha < \gamma$. Then for a given κ we see that we can find a family $(\kappa_\alpha)_{\alpha \leq \beta}$ of cardinal numbers such that $\kappa_0 = \kappa$ and such that if $(Y_\alpha)_{\alpha \leq \beta}$ is constructed as above where $|Y| = |Y_0| \leq \kappa$ then $|Y_\alpha| \leq \kappa_\alpha$ for each $\alpha \leq \beta$.

We now note that if $S \subset X$ and if $S \rightarrow Y_\beta$ is such that there is a factorization $S \rightarrow Y_\alpha \rightarrow Y_\beta$ for some $\alpha < \beta$ then $S \rightarrow Y_\alpha$ has an extension $X \rightarrow Y_{\alpha+1}$ so giving the extension $X \rightarrow Y_{\alpha+1} \rightarrow Y_\beta$ of the original $S \rightarrow Y_\beta$. So we must choose β such that every such $S \rightarrow Y_\beta$ has such a factorization. So we want to argue that we can find a β independent of Y so that the corresponding Y_β will always be injective. But this again just uses Baer's original idea and appeals to the fact that every object is small relative to the class of monomorphisms (see [22, pg. 32] for the terminology and [1, Proposition 2.2] for an argument). The object in question would be the coproduct of a representative set of subobjects S of X . \square

Lemma 2.11. *Given $n \geq 0$ and a cardinal κ there is a cardinal λ so that if L is an object of injective dimension at most n and if $S \subset L$ is such that $|S| \leq \kappa$ then there is an $L' \subset L$ such that $S \subset L'$, such that $|L'| \leq \lambda$ and such that both L' and L/L' have injective dimension at most n .*

Proof. The proof is just the categorical version of [2, Proposition 2.5]. There the argument uses Corollary 2.3 and Lemma 2.4. To get our version of Corollary 2.3 we only need use the Baer criterion with respect to a generator X of \mathcal{A} . The categorical version of Lemma 2.4 is just the preceding lemma. \square

Definition 2.12. A class \mathcal{L} of objects of \mathcal{A} is said to be a Kaplansky class (see [19]) of \mathcal{A} if for each cardinal κ there is a cardinal λ such that if $S \subset L$ for some $L \in \mathcal{L}$ where $|S| \leq \kappa$ then there is an $L' \subset L$ with $S \subset L'$ where $|L'| \leq \lambda$ and where L' and L/L' are both in \mathcal{L} .

Corollary 2.13. If $n \geq 0$ and if \mathcal{L} is the class of objects L of \mathcal{A} such that $\text{injdim} L \leq n$ then \mathcal{L} is a Kaplansky class of \mathcal{A} .

Proof. Immediate from the above. \square

In what follows we will no longer need to refer to a fixed generator X of \mathcal{A} and so will use the symbol X to stand for an arbitrary object of \mathcal{A} .

Corollary 2.14. If $n \geq 0$ and if \mathcal{L} is the class of objects L of \mathcal{A} such that $\text{injdim} L \leq n$ then \mathcal{L} is preenveloping.

Proof. Given the object X of \mathcal{A} we consider morphisms $X \rightarrow L$ where $L \in \mathcal{L}$. Using Lemmas 2.5, 2.6 and the Corollaries 2.4 and 2.13 we see that we can find a cardinal λ such that for any $X \rightarrow L$ with $L \in \mathcal{L}$ there is a factorization $X \rightarrow L' \subseteq L$ with $L' \in \mathcal{L}$ and $|L'| \leq \lambda$. Then using Corollary 2.4 we see that there is a set \mathcal{L}^* of objects in \mathcal{L} such that if $L \in \mathcal{L}$ and if $|L| \leq \lambda$ then $L \cong L^*$ for some $L^* \in \mathcal{L}^*$. So noting that for each $L^* \in \mathcal{L}^*$, $\text{Hom}(X, L^*)$ is a set we see that we can find a family $(\sigma_i)_{i \in I}$ of morphisms $\sigma_i : X \rightarrow L_i$ with $L_i \in \mathcal{L}^*$ so that if $L^* \in \mathcal{L}^*$ and if $\sigma : X \rightarrow L^*$ is a morphism, then $\sigma = \sigma_i$ for some $i \in I$. Then the morphism $X \rightarrow \prod_{i \in I} L_i$ given by the family $(\sigma_i)_{i \in I}$ is the desired preenvelope. \square

We remark that for any X of \mathcal{A} we have $X \subseteq E$ for an injective object E of \mathcal{C} . Since $E \in \mathcal{L}$ we have the factorization $X \rightarrow L \rightarrow E$ for any such \mathcal{L} -preenvelope $X \rightarrow L$. Hence $X \rightarrow L$ is necessarily a monomorphism.

We will eventually want another property of a class \mathcal{L} of \mathcal{A} . In the next Lemma we will use the notion of transfinite extensions. For a definition see ([23, Section 6]).

We recall that since \mathcal{A} is a Grothendieck category (and so has enough injectives) we can define the functors $\text{Ext}^n(X, Y)$ for all $n \geq 0$.

Definition 2.15. If X is an object of \mathcal{A} we say that $\text{projdim} X \leq n$ if

$$\text{Ext}^i(X, -) = 0 \text{ for } i \geq n + 1.$$

Analogously if an object X has finite injective dimension at most n we write $\text{injdim} X \leq n$. So then we define $\text{projdim} X$ and $\text{injdim} Y$ as usual. We define $\text{FPD}(\mathcal{A})$ as the supremum of $\text{projdim} X$ over all X such that $\text{projdim} X < \infty$. We define $\text{FID}(\mathcal{A})$ in a similar manner.

Definition 2.16. A class of objects \mathcal{L} of \mathcal{A} is said to be closed under transfinite extensions if whenever $(L_\alpha)_{\alpha \leq \lambda}$ is a continuous chain of objects of \mathcal{A} such that $L_0 = 0$ and such that $L_{\alpha+1}/L_\alpha \in \mathcal{L}$ whenever $\alpha + 1 \leq \lambda$ we also have L_λ in \mathcal{L} .

Lemma 2.17. Given $n \geq 0$, if \mathcal{L} is the class of objects L of \mathcal{A} such that $\text{projdim} L \leq n$ then \mathcal{L} is closed under transfinite extensions.

Proof. This result (for modules) is due to Auslander. Our result follows from Hovey's proof of [23, Lemma 6.2]. In that proof he shows that for any object Y of \mathcal{A} the class of X such that $\text{Ext}^1(X, Y) = 0$ is closed under transfinite extensions (this is a categorical version of a theorem of Eklof [9]). If we let Y range through the class of n -th cosyzygies of objects of \mathcal{A} we get the result by using the fact that $\text{Ext}^{n+1}(X, Z) = \text{Ext}^1(X, Y)$ if Y is such a cosyzygy for Z . \square

Definition 2.18. We will say that \mathcal{A} is a Gorenstein category if the following hold:

- 1) For any object L of \mathcal{A} , $\text{projdim } L < \infty$ if and only if $\text{injdim } L < \infty$.
- 2) $\text{FPD}(\mathcal{A}) < \infty$ and $\text{FID}(\mathcal{A}) < \infty$.
- 3) \mathcal{A} has a generator L such that $\text{projdim } L < \infty$.

So now when we say that $(\mathcal{A}, \mathcal{L})$ is a Gorenstein category we mean that \mathcal{A} is such a category and that \mathcal{L} is the class of objects L of \mathcal{A} such that $\text{projdim } L < \infty$.

If furthermore $\text{FPD}(\mathcal{A}) \leq n$ and $\text{FID}(\mathcal{A}) \leq n$ we will say that $(\mathcal{A}, \mathcal{L})$ has dimension at most n and then define the dimension of $(\mathcal{A}, \mathcal{L})$ to be the least such n .

Remark. Generators with finite projective dimension in Grothendieck categories without enough projectives were also considered by Hovey in [24, Section 2].

Definition 2.19. By a complete projective resolution in \mathcal{A} we mean a complex $\mathbf{P} = (P^n)$ for $n \in \mathbb{Z}$ (so the complex is infinite in both directions) of projective objects such that \mathbf{P} is exact and such that the complex $\text{Hom}(\mathbf{P}, Q)$ is also exact for any projective object Q of \mathcal{A} . If C is the kernel of $P^0 \rightarrow P^1$ then we say that \mathbf{P} is a complete projective resolution of C . An object C is said to be Gorenstein projective if it admits such a complete projective resolution. Complete injective resolutions and Gorenstein injective objects are defined dually (see [16]).

Definition 2.20. Given a class \mathcal{C} of objects of \mathcal{A} then the class of objects Y of \mathcal{A} such that $\text{Ext}^1(C, Y) = 0$ for all $C \in \mathcal{C}$ is denoted \mathcal{C}^\perp . Similarly ${}^\perp\mathcal{C}$ denotes the class of X such that $\text{Ext}^1(X, C) = 0$ for all $C \in \mathcal{C}$.

Proposition 2.21. Let \mathcal{A} be a Grothendieck category and let $\mathcal{C}(\mathcal{J})$ be the class of Gorenstein projective (injective) objects of \mathcal{A} (there may not be any other than 0). Then $\mathcal{C}^\perp ({}^\perp\mathcal{J})$ is a thick subcategory of \mathcal{A} containing all injective and all projective objects of \mathcal{A} . So $\mathcal{C}^\perp ({}^\perp\mathcal{J})$ contains all objects of finite injective dimension and all objects having a finite projective resolution.

Proof. We prove the result for \mathcal{C} and \mathcal{C}^\perp . A dual argument gives the result for \mathcal{J} and ${}^\perp\mathcal{J}$.

We note that $Y \in \mathcal{C}^\perp$ if and only if for every complete projective resolution \mathbf{P} the complex $\text{Hom}(\mathbf{P}, Y)$ is exact. So easily \mathcal{C}^\perp is closed under retracts.

If $0 \rightarrow Y' \rightarrow Y \rightarrow Y'' \rightarrow 0$ is an exact sequence in \mathcal{A} and if $\mathbf{P} = (P^n)$ is a complete projective resolution, then since each P^n is projective we get that

$$0 \rightarrow \text{Hom}(\mathbf{P}, Y') \rightarrow \text{Hom}(\mathbf{P}, Y) \rightarrow \text{Hom}(\mathbf{P}, Y'') \rightarrow 0$$

is an exact sequence of complexes. Hence if any two of these complexes is exact, so is the third. Hence \mathcal{C}^\perp is a thick subcategory of \mathcal{A} .

If E is an injective object of \mathcal{A} , then since any complete projective resolution \mathbf{P} is exact, $\text{Hom}(\mathbf{P}, E)$ is also exact. So $E \in \mathcal{C}^\perp$. If Q is a projective object of \mathcal{A} , then by the definition of a complete projective resolution \mathbf{P} the complex $\text{Hom}(\mathbf{P}, Q)$ is exact. Hence we get all the claims about \mathcal{C}^\perp . \square

Definition 2.22. If \mathcal{A} is Grothendieck with enough projectives define $Gpd(X)$ (the Gorenstein projective dimension of X) in the usual way. That is, $Gpd(X) = n$ if the first syzygy of X that is Gorenstein projective is the n -th one and $Gpd(X) = \infty$ if there is no such syzygy. Then define $glGpd(\mathcal{A})$ (the global Gorenstein projective dimension of \mathcal{A}). Then also define $Gid(Y)$ and $glGid(\mathcal{A})$ (without assuming \mathcal{A} has enough projectives).

Definition 2.23. A pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of \mathcal{A} is said to be cotorsion pair if $\mathcal{F}^\perp = \mathcal{C}$ and if ${}^\perp\mathcal{C} = \mathcal{F}$. It is said to be complete if for each X and Y of \mathcal{A} there exist exact sequences $0 \rightarrow C \rightarrow F \rightarrow X \rightarrow 0$ and $0 \rightarrow Y \rightarrow C' \rightarrow F' \rightarrow 0$ where $F, F' \in \mathcal{F}$ and where $C, C' \in \mathcal{C}$. We furthermore say that $(\mathcal{F}, \mathcal{C})$ is functorially complete if these sequences can be chosen in a functorial manner (depending on X and on Y) (see Definition 2.3 of [23]).

We now give our main result.

Theorem 2.24. *If $(\mathcal{A}, \mathcal{L})$ is a Gorenstein category then $(\mathcal{L}, \mathcal{L}^\perp)$ is a complete and hereditary cotorsion pair on \mathcal{A} and \mathcal{L}^\perp is the class of Gorenstein injective objects of \mathcal{A} . If $(\mathcal{A}, \mathcal{L})$ has dimension at most n then $Gid(Y) \leq n$ for all objects Y of \mathcal{A} .*

Proof. To get that $(\mathcal{L}, \mathcal{L}^\perp)$ is a cotorsion pair we only need argue that ${}^\perp(\mathcal{L}^\perp) = \mathcal{L}$. Clearly $\mathcal{L} \subset {}^\perp(\mathcal{L}^\perp)$. But since $\text{projdim } L \leq n$ for all $L \in \mathcal{L}$, \mathcal{L}^\perp contains all the n -th cosyzygies Y of objects of \mathcal{A} . If $\text{Ext}^1(L, Y) = 0$ for all such Y then $\text{projdim } L \leq n$ and so ${}^\perp(\mathcal{L}^\perp) \subset \mathcal{L}$ and so we get $\mathcal{L} = {}^\perp(\mathcal{L}^\perp)$.

As a first step toward arguing that this cotorsion pair is complete, we want to argue that it is cogenerated by a set, i.e. there is a set \mathcal{S} with $\mathcal{S} \subseteq \mathcal{L}$ such that $\mathcal{S}^\perp = \mathcal{L}^\perp$. But this follows from the fact that \mathcal{L} is closed under transfinite extensions and that it is a Kaplansky class (see Lemma 2.17 and Corollary 2.13 respectively). For let κ be $\sup|T|$ where T is the image of an arbitrary morphism $X \rightarrow L$ for $L \in \mathcal{L}$ (every such $T \cong X/Z$ for some $Z \subseteq X$, so we are using Corollary 2.3). Now let λ be as in Definition 2.12 for this κ and \mathcal{L} . Now let \mathcal{S} be a set of representatives of $S \in \mathcal{L}$ such that $|S| \leq \lambda$ (here we are using Corollary 2.4 with κ replaced by λ). Then we see that every $L \in \mathcal{L}$ can be written as the union of a continuous chain $(L_\alpha)_{\alpha \leq \beta}$ (for some ordinal β) of subobjects such that $\alpha + 1 \leq \beta$, $L_{\alpha+1}/L_\alpha$ is isomorphic to an $S \in \mathcal{S}$. Then we appeal to [10, Lemma 1] to see that $\mathcal{S}^\perp = \mathcal{L}^\perp$.

So now to get functorial completeness we want to appeal to [23, Theorem 6.5]. To do so we need to show that $(\mathcal{L}, \mathcal{L}^\perp)$ is small according to [23, Definition 6.4]. In this definition, Hovey gives three conditions on a cotorsion pair in a Grothendieck category that are required for it to be small. In our situation the cotorsion pair is $(\mathcal{L}, \mathcal{L}^\perp)$. Applied to this pair (and, again, in our situation) Hovey's conditions are: *i)* \mathcal{L} contains a set of generators of the category, *ii)* $(\mathcal{L}, \mathcal{L}^\perp)$ is cogenerated by a set $\mathcal{S} \subset \mathcal{L}$, *iii)* for each $L \in \mathcal{S}$ there is a given exact sequence $0 \rightarrow K \rightarrow U \rightarrow L \rightarrow 0$

such that for any Y , $Y \in \mathcal{L}^\perp$ if and only if $\text{Hom}(U, Y) \rightarrow \text{Hom}(K, Y) \rightarrow 0$ is exact for all such exact sequence.

We have condition *i*) by our definition of a Gorenstein category. We have *ii*) from the above.

We now argue that $(\mathcal{L}, \mathcal{L}^\perp)$ satisfies a slightly weaker version of *iii*). We argue that for each $L \in \mathcal{S}$ we have a set of exact sequences $0 \rightarrow K \rightarrow U \rightarrow L \rightarrow 0$ (one set for each $L \in \mathcal{S}$) so that $Y \in \mathcal{L}^\perp$ if and only if $\text{Hom}(U, Y) \rightarrow \text{Hom}(K, Y) \rightarrow 0$ is exact for all such exact sequences. Our set for a given L will be a set of representatives of all short exact sequences $0 \rightarrow K \rightarrow U \rightarrow L \rightarrow 0$ where $|U| \leq \kappa$ and where we get the κ from Corollary 2.3 when we let the Y of that lemma be our L .

So now suppose G is an object such that $\text{Hom}(U, G) \rightarrow \text{Hom}(K, G) \rightarrow 0$ is exact for all of the exact sequences in our set. We want to argue then that $\text{Ext}^1(L, G) = 0$ for all $L \in \mathcal{L}$. Because \mathcal{S} cogenerates $(\mathcal{L}, \mathcal{L}^\perp)$ it suffices to argue that $\text{Ext}^1(L, G) = 0$ for all $L \in \mathcal{S}$. So let $0 \rightarrow G \rightarrow V \rightarrow L \rightarrow 0$ be exact with $L \in \mathcal{S}$. We want to argue that this sequence splits. But we know that there is a $U \subset V$ such that $|U| \leq \kappa$ and such that $U \rightarrow L$ is an epimorphism. Then, up to isomorphism, we can suppose that with $K = G \cap U$, $0 \rightarrow K \rightarrow U \rightarrow L \rightarrow 0$ is one of our sequences.

So consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & U & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & G & \longrightarrow & V & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

Since by hypothesis $K \rightarrow G$ can be extended to $U \rightarrow G$ we get that the bottom sequence splits.

If Z is a Gorenstein injective object then using a complete injective resolution of Z it is easy to argue that $\text{Ext}^1(L, Z) = 0$ when $\text{projdim } L \leq n$. For such a Z is an n -th cosyzygy of some object W of \mathcal{A} and $\text{Ext}^1(L, Z) = \text{Ext}^{n+1}(L, W) = 0$. Hence \mathcal{L}^\perp contains all the Gorenstein injective objects of \mathcal{A} .

We now argue that if $G \in \mathcal{L}^\perp$ then G is Gorenstein injective. If $0 \rightarrow G \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ is an injective resolution of G , then since $E \in \mathcal{L}$ for any injective object E we get that $\text{Hom}(E, -)$ applied to $0 \rightarrow G \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$ gives an exact sequence. So we have the right half of a complete injective resolution of G . We must now show that we can construct the left half. Since $(\mathcal{L}, \mathcal{L}^\perp)$ is complete there is an exact sequence $0 \rightarrow K \rightarrow L \rightarrow G \rightarrow 0$ with $K \in \mathcal{L}^\perp$ and with $L \in \mathcal{L}$. So $L \rightarrow G$ is a special \mathcal{L} -precover of G . Let $L \subset E$ where E is injective. Then $\text{injdim } E/L < \infty$ and so $E/L \in \mathcal{L}$ and $\text{Ext}^1(E/L, G) = 0$. This means that $L \rightarrow G$ can be extended to (necessarily epimorphic) $E \rightarrow G$. Let $0 \rightarrow H \rightarrow E \rightarrow G \rightarrow 0$ be exact. Then $E \rightarrow G$ is also an \mathcal{L} -precover. So $\text{Hom}(\overline{E}, E) \rightarrow \text{Hom}(\overline{E}, G) \rightarrow 0$ is exact for any injective \overline{E} . We now see that $\text{Ext}^1(\overline{E}, H) = 0$. This follows from the exact $0 \rightarrow \text{Hom}(\overline{E}, H) \rightarrow \text{Hom}(\overline{E}, E) \rightarrow$

$\text{Hom}(\overline{E}, G) \rightarrow \text{Ext}^1(\overline{E}, H) \rightarrow \text{Ext}^1(\overline{E}, E) = 0$ and the fact that $\text{Hom}(\overline{E}, E) \rightarrow \text{Hom}(\overline{E}, G)$ is surjective. So now replace G with H in the argument above and we find an analogous $E' \rightarrow H$ with E' injective and such that this morphism is an \mathcal{L} -precover of H . Since we can continue this procedure we see that we can construct a complete injective resolution

$$\cdots \rightarrow E' \rightarrow E \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of G . Finally we want to argue that $\text{Gid}(Y) \leq n$ for any object Y of \mathcal{A} . But if Z is an n -th cosyzygy of Y we have $\text{Ext}^1(L, Z) = 0$ for all $L \in \mathcal{L}$. Hence Z is Gorenstein injective.

We now argue that our cotorsion pair $(\mathcal{L}, \mathcal{L}^\perp)$ is hereditary. In this situation this means that we need argue that if $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ is exact with $G', G \in \mathcal{L}^\perp$, then $G'' \in \mathcal{L}^\perp$. So we need to argue that G'' is Gorenstein injective. We let $L \rightarrow G$ be a special \mathcal{L} -precover of G . This means we have an exact sequence

$$0 \rightarrow H \rightarrow L \rightarrow G \rightarrow 0$$

with H Gorenstein injective. So we get a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & H & \xlongequal{\quad} & H & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{G} & \longrightarrow & L & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. Then since \mathcal{L}^\perp is closed under extensions we get that $L \in \mathcal{L}^\perp$. So $L \in \mathcal{L}^\perp \cap \mathcal{L}$. Considering an exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$$

with E injective we get by Proposition 2.21 that $M \in \mathcal{L}$. So the sequence splits and we have L injective. We have the exact $0 \rightarrow \overline{G} \rightarrow L \rightarrow G'' \rightarrow 0$ with \overline{G} Gorenstein injective and where $\text{Hom}(E, L) \rightarrow \text{Hom}(E, G'') \rightarrow 0$ is exact if E is injective, since $\text{Ext}^1(E, \overline{G}) = 0$ by Proposition 2.21. So using the left half of a complete injective resolution of \overline{G} along with the exact $0 \rightarrow \overline{G} \rightarrow L \rightarrow G'' \rightarrow 0$ and an injective resolution of G'' we get a complete injective resolution of G'' . \square

Theorem 2.25. *If $(\mathcal{A}, \mathcal{L})$ be a Gorenstein category of dimension at most n having enough projectives. Then for an object C of \mathcal{A} the following are equivalent:*

- 1) C is an n -th syzygy.
- 2) $C \in {}^\perp \mathcal{L}$.
- 3) C is Gorenstein projective.

As a consequence we get that $glGpd(\mathcal{A}) \leq n$ and that $({}^\perp \mathcal{L}, \mathcal{L})$ is a complete an hereditary cotorsion pair.

Proof. 1) \Rightarrow 2) If C is an n -th syzygy of X then $\text{Ext}^1(C, L) = \text{Ext}^{n+1}(X, L)$ for any object L of \mathcal{A} . If $L \in \mathcal{L}$ then $\text{injdim} L \leq n$ so $\text{Ext}^{n+1}(X, L) = 0$. Hence $\text{Ext}^1(C, L) = 0$ and so $C \in {}^\perp \mathcal{L}$.

2) \Rightarrow 3) By Corollary 2.14 we know C has an \mathcal{L} -preenvelope $C \rightarrow L$. As noted after the proof of that Corollary, $C \rightarrow L$ is a monomorphism. Let $0 \rightarrow L' \rightarrow P \rightarrow L \rightarrow 0$ be exact where P is projective. Then since $\text{projdim} L < \infty$ we get $\text{projdim} L' < \infty$ and so $L' \in \mathcal{L}$. Since $\text{Ext}^1(C, L') = 0$, $C \rightarrow L$ can be factored $C \rightarrow P \rightarrow L$. Then $C \rightarrow P$ is a monomorphism and is also an \mathcal{L} -preenvelope of C . So continuing this procedure we get that we get an exact sequence

$$0 \rightarrow C \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \dots$$

where each P^n is projective and such that if Q is projective then the functor $\text{Hom}(-, Q)$ leaves the sequence exact.

Now let

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow C \rightarrow 0$$

be a projective resolution of C . Since $\text{Ext}^i(C, Q) = 0$ for $i \geq 1$ and Q projective we see that $\text{Hom}(-, Q)$ leaves this sequence exact. Consequently pasting we see that we get a complete projective resolution of C .

3) \Rightarrow 1) is trivial.

The equivalent 1) \Leftrightarrow 3) gives that $glGpd(\mathcal{A}) \leq n$.

We know argue that $({}^\perp \mathcal{L}, \mathcal{L})$ is a complete cotorsion pair. The fact that $glGpd(\mathcal{A}) \leq n$ gives that for each object X of \mathcal{A} there is an exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow X \rightarrow 0$$

with C Gorenstein projective and $\text{projdim} L \leq n - 1$ in case $n \geq 1$ (and with $L = 0$ if $n = 0$). The argument is essentially the dual to the proof of [17, Theorem 11.2.1].

Then using what is called the Salce trick (see [17, Proposition 7.1.7]) we get that for every object X of \mathcal{A} there is an exact sequence $0 \rightarrow X \rightarrow L \rightarrow C \rightarrow 0$ with $L \in \mathcal{L}$ and $C \in {}^\perp \mathcal{L}$. Hence if $X \in ({}^\perp \mathcal{L})^\perp$ the sequence splits. So X as a summand of $L \in \mathcal{L}$ is also in \mathcal{L} . So we get that $({}^\perp \mathcal{L})^\perp = \mathcal{L}$ and that $({}^\perp \mathcal{L}, \mathcal{L})$ is a cotorsion pair. then completeness follows from the above. Since an exact $0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$ with $L, L'' \in \mathcal{L}$ gives $L' \in \mathcal{L}$ and since \mathcal{A} has enough projectives we get that $({}^\perp \mathcal{L}, \mathcal{L})$ is hereditary. \square

Lemma 2.26. *If \mathcal{A} is Grothendieck with enough projectives then*

$$glGpd(\mathcal{A}) \leq m \Rightarrow FID(\mathcal{A}) \leq m$$

and the converse holds if \mathcal{A} is Gorenstein (so $glGpd(\mathcal{A}) = FID(\mathcal{A})$ in this case). Dually we have that $glGid(\mathcal{A}) \leq k \Rightarrow FPD(\mathcal{A}) \leq k$ and the converse holds if \mathcal{A} is Gorenstein (so $glGid(\mathcal{A}) = FPD(\mathcal{A})$ in this case).

Proof. Let L have finite injective dimension. Now, given any object X of \mathcal{A} , let $0 \rightarrow C \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow X \rightarrow 0$ be a partial projective resolution of X . Then C is Gorenstein projective and so $\text{Ext}^1(C, L) = 0$. This gives that $\text{Ext}^{m+1}(X, L) = 0$. Since X was arbitrary, $\text{injdim } L \leq m$. Now assume $FID(\mathcal{A}) \leq m$ and that \mathcal{A} is Gorenstein. Let $0 \rightarrow C \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow X \rightarrow 0$ be a partial projective resolution of any X . Then if $L \in \mathcal{L}$ we have $\text{injdim } L < \infty$ so $\text{injdim } L \leq m$. So $\text{Ext}^{m+1}(X, L) = 0$, i.e., $\text{Ext}^1(C, L) = 0$. So $C \in {}^\perp \mathcal{L}$. But this means C is Gorenstein projective by Theorem 2.25. The argument for the rest of the proof is dual to this argument. \square

Theorem 2.27. *Let \mathcal{A} be a Grothendieck category with enough projectives. Then the following are equivalent:*

- 1) \mathcal{A} is Gorenstein.
 - 2) $glGpd(\mathcal{A}) < \infty$ and $glGid(\mathcal{A}) < \infty$.
- Moreover, if (1) (or (2)) holds we have

$$FID(\mathcal{A}) = FPD(\mathcal{A}) = glGpd(\mathcal{A}) = glGid(\mathcal{A}).$$

Proof. We have 1) implies 2) by Theorem 2.24 and Theorem 2.25.

So now assume 2). Let \mathcal{L} consist of all object L with $\text{projdim } L < \infty$ and let \mathcal{L}' consist of all objects L' with $\text{injdim } L' < \infty$. Now assume $glGpd(\mathcal{A}) \leq n$ and that $glGid(\mathcal{A}) \leq n$. If C is an n -th syzygy of an object X of \mathcal{A} then C is Gorenstein projective. So by Proposition 2.21, $\text{Ext}^1(C, L) = 0$ for any $L \in \mathcal{L}$. So $\text{Ext}^{n+1}(X, L) = 0$ for any X and any $L \in \mathcal{L}$. So $\text{injdim } L \leq n$ and so $L \in \mathcal{L}'$. A dual argument gives that $\mathcal{L}' \subseteq \mathcal{L}$ and that if $L' \in \mathcal{L}'$ then $\text{projdim } L' \leq n$. So we get that $\mathcal{L}' = \mathcal{L}$ and so that $(\mathcal{A}, \mathcal{L})$ is a Gorenstein category.

To get the desired equality we note we have $FPD(\mathcal{A}) = glGid(\mathcal{A})$ and $FID(\mathcal{A}) = glGpd(\mathcal{A})$ by Lemma 2.26. So we argue that $glGid(\mathcal{A}) = glGpd(\mathcal{A})$. We use the fact that $({}^\perp \mathcal{L}, \mathcal{L})$ and $(\mathcal{L}, \mathcal{L}^\perp)$ are complete cotorsion pairs with ${}^\perp \mathcal{L} = \mathcal{C}$ the class of Gorenstein projectives and $\mathcal{L}^\perp = \mathcal{J}$ the class of Gorenstein injectives.

Furthermore we know $\text{Hom}(-, -)$ is right balanced by $\mathcal{C} \times \mathcal{J}$ (see [20, Theorem 1.2.19]). So we can define relative derived functors of $\text{Hom}(-, -)$. These are denoted by $\text{Gext}^n(X, Y)$ ($n \geq 0$ for any X, Y objects in \mathcal{A}). Once we have this machinery, then we use the usual argument that the following are equivalent for an abelian category \mathcal{A} with enough injectives and projectives and an n with $0 \leq n < \infty$:

- 1) $\text{projdim } X \leq n$ for all X ,
- 2) $\text{injdim } Y \leq n$ for all Y ,
- 3) $\text{Gext}^i(-, -) = 0$ for $i \geq n + 1$.

\square

We get examples of Gorenstein categories with enough projectives by considering ${}_R \text{Mod}$ where R is an (Iwanaga) Gorenstein ring (see 9.1 of [17]). These rings are noetherian. But there are many nontrivial nonnoetherian R such that

${}_R\text{Mod}$ is Gorenstein (see [13] and [14]). By a trivial example we mean one with $\text{l.gldim } R < \infty$.

Proposition 2.28. *If \mathcal{A} is a Gorenstein category of dimension at most n , then for every object Y there is an exact sequence*

$$0 \rightarrow Y \rightarrow G \rightarrow L \rightarrow 0$$

where G is Gorenstein injective and where $\text{injdim } L \leq n - 1$.

Proof. We can mimic the proof for modules given in [17, Theorem 11.2.1]. \square

We note that if $0 \rightarrow Y \rightarrow G \rightarrow L \rightarrow 0$ is as above, then $Y \rightarrow G$ is a special Gorenstein injective preenvelope of Y . Using these preenvelopes we get a version of relative homological algebra that is called Gorenstein homological algebra. We see that for each object Y we get a Gorenstein injective resolution of Y . This just means an exact sequence

$$0 \rightarrow Y \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$$

where all the G^n are Gorenstein injective and such that $\text{Hom}(-, G)$ leaves the sequence exact whenever G is Gorenstein injective. Such a complex is unique up to homotopy and can be used to give right derived functors $\text{Gext}^i(X, Y)$ of Hom (these functors were introduced in [15] and were later studied in [6] with different notation). There are obvious natural maps $\text{Gext}^i(X, Y) \rightarrow \text{Ext}^i(X, Y)$ for all $i \geq 0$.

The Tate cohomology functors $\widehat{\text{Ext}}_{\mathcal{A}}^i(X, Y)$ (for any $i \in \mathbb{Z}$) are defined as follows.

Let $0 \rightarrow Y \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{n-1} \rightarrow G \rightarrow 0$ be a partial injective resolution of Y . Then G is Gorenstein injective so has a complete injective resolution which we can take to be

$$\mathbf{E} = \dots \rightarrow E^{n-1} \rightarrow E^n \rightarrow E^{n+1} \rightarrow \dots$$

with $G = \text{Ker}(E^n \rightarrow E^{n+1})$. This complex is unique up to homotopy and so we define the groups $\widehat{\text{Ext}}^i(X, Y)$ to be the i -th cohomology groups of the complex $\text{Hom}(X, \mathbf{E})$.

If $0 \rightarrow Y \rightarrow E'^0 \rightarrow E'^1 \rightarrow \dots$ is an injective resolution of Y then there is a commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & E'^0 & \longrightarrow & E'^1 & \longrightarrow & E'^2 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & E^{-2} & \longrightarrow & E^{-1} & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \cdots \end{array}$$

The associated map of complexes is unique up to homotopy and gives rise to natural maps

$$\text{Ext}^i(X, Y) \rightarrow \widehat{\text{Ext}}^i(X, Y)$$

for $i \geq 0$. In [6], Avramov and Martsinkovsky gave a beautiful result relating these two collections of natural maps. In their situation they considered finitely

generated modules M with $Gpd(M) < \infty$ over a noetherian ring. And so used a complete projective resolution as above to get the Tate cohomology functors. A. Iacob in [26] removed the finitely generated assumption and also showed how to use the hypothesis $Gid(Y) < \infty$ instead of the Gorenstein projective dimension hypothesis. Then she showed that if both $Gpd(X) < \infty$ and $Gid(Y) < \infty$ then the two procedures give the same groups $\widehat{\text{Ext}}^i(X, Y)$. So due to Iacob's extension of the Avramov and Martsinkovsky results we get:

Proposition 2.29. *If \mathcal{A} is a Gorenstein category of dimension at most n then for all objects X and Y of \mathcal{A} there exist natural exact sequences*

$$0 \rightarrow \text{Gext}^1(X, Y) \rightarrow \text{Ext}^1(X, Y) \rightarrow \widehat{\text{Ext}}^1(X, Y) \rightarrow \text{Gext}^2(X, Y) \rightarrow \cdots \rightarrow \text{Gext}^n(X, Y) \rightarrow \text{Ext}^n(X, Y) \rightarrow \widehat{\text{Ext}}^n(X, Y) \rightarrow 0.$$

The following is immediate:

Proposition 2.30. *In the situation above, the following are equivalent:*

- 1) $\text{Gext}^1(X, Y) \rightarrow \text{Ext}^1(X, Y)$ is an isomorphism for all X, Y .
- 2) $\text{Gext}^i(X, Y) \rightarrow \text{Ext}^i(X, Y)$ is an isomorphism for all X, Y and all $i \geq 1$.
- 3) $\widehat{\text{Ext}}^i(X, Y) = 0$ for all X, Y and $i \geq 1$.
- 4) $\mathcal{L} = \mathcal{A}$.
- 5) $\mathcal{J} = \mathcal{I}$ (\mathcal{I} is the class of injective objects).
- 6) $\mathcal{C} = \mathcal{P}$ (\mathcal{P} is the class of projective objects, only in case \mathcal{A} has enough projectives).
- 7) $\widehat{\text{Ext}}^i(X, Y) = 0$ for all X, Y and $i \in \mathbb{Z}$.
- 8) For a fixed i , $\widehat{\text{Ext}}^i(X, Y) = 0$ for all X, Y .

Proof. This is just an application of [23, Theorem 2.2], (also see [23, Theorem 8.6]). For the first claim we use the notation of that paper and let $\mathcal{C} = \mathcal{A}$, $\mathcal{W} = \mathcal{L}$ and $\mathcal{F} = \mathcal{L}^\perp$. For the second part let $\mathcal{C} = {}^\perp \mathcal{L}$, $\mathcal{W} = \mathcal{L}$ and $\mathcal{F} = \mathcal{A}$. \square

3. THE CATEGORY OF QUASI-COHERENT SHEAVES OVER $\mathbf{P}^n(A)$

Let A be a commutative noetherian ring. This section deals with the category of quasi-coherent sheaves over $\mathbf{P}^n(A)$ and then over closed subschemes $X \subset \mathbf{P}^n(A)$. We want to prove that for certain such X this category is Gorenstein. In this section we will use the fact that $\Omega\text{co}(X)$ over any scheme X is equivalent to the category of certain representations of some quiver Q , which may be chosen in various ways, (see [11, Section 2] or [25, 3.1] for an explanation of this viewpoint). In case $X \subset \mathbf{P}^n(A)$ is a closed subscheme, this quiver can be taken to have an especially nice form. First, $\mathbf{P}^n(A)$ and $\Omega\text{co}(\mathbf{P}^n(A))$ can be associated with the quiver Q whose vertices are the subsets

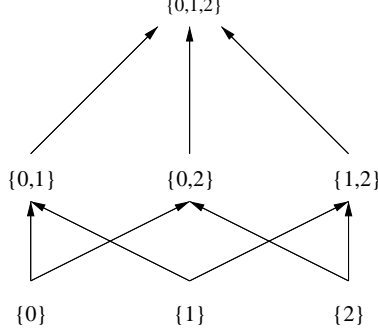
$$v \subseteq \{0, 1, 2, \dots, n\}, \quad v \neq \emptyset,$$

where there is a unique arrow $v \rightarrow w$ when $v \subseteq w$. The associated ring \mathfrak{R} is such that $\mathfrak{R}(v)$ is the ring of polynomials with coefficients in A in the variables x_i/x_j , where $0 \leq i \leq n$ and $j \in v$. Then, when $v \subseteq w$, $\mathfrak{R}(v) \rightarrow \mathfrak{R}(w)$ is just a localization (by the multiplicative set generated by the x_i/x_j , $i \in w, j \in v$).

Example. If we consider the projective scheme $\mathbf{P}^1(A)$, the previous quiver is

$$\{0\} \hookrightarrow \{0, 1\} \hookleftarrow \{1\}$$

Let us consider the projective scheme $\mathbf{P}^2(A)$. Then the corresponding quiver has the form



Notice that, for example, we may delete the arrow from $\{0\}$ to $\{0, 1, 2\}$ because this map is the obvious composition $\{0\} \hookrightarrow \{0, 1\} \hookrightarrow \{0, 1, 2\}$

A closed subscheme $X \subseteq \mathbf{P}^n(A)$ is given by a quasi-coherent sheaf of ideals, i.e. we have an ideal $\mathfrak{I}(v) \subseteq \mathfrak{R}(v)$ for each v with

$$\mathfrak{R}(w) \otimes_{\mathfrak{R}(v)} \mathfrak{I}(v) \cong \mathfrak{I}(w),$$

when $v \subseteq w$. This just means $\mathfrak{I}(v) \rightarrow \mathfrak{I}(w)$ is the localization of $\mathfrak{I}(v)$ by the same multiplicative set as above. But then

$$\frac{\mathfrak{R}}{\mathfrak{I}}(v) = \frac{\mathfrak{R}(v)}{\mathfrak{I}(v)} \rightarrow \frac{\mathfrak{R}(w)}{\mathfrak{I}(w)}$$

(when $v \subseteq w$) is also a localization. So now, to simplify the notation, we will use $\mathfrak{R}(v)$ in place of $\frac{\mathfrak{R}(v)}{\mathfrak{I}(v)}$ to give the \mathfrak{R} associated with X .

The next result is standard in algebraic geometry. Those who work in this area will recognize our D^v as essentially the i_* of [21, Proposition II.5.8].

Proposition 3.1. *For a given vertex v , the functor*

$$H^v : \mathfrak{Qco}(X) \rightarrow \mathfrak{R}(v)\text{-Mod}$$

given by $H^v(M) = M(v)$ has a right adjoint.

Proof. We consider v as fixed. Let N be an $\mathfrak{R}(v)$ -module. We construct a quasi-coherent \mathfrak{R} -module $D^v(N)$ as follows: for any w let $D^v(N)(w) = \mathfrak{R}(v \cup w) \otimes_{\mathfrak{R}(v)} N$ (so $D^v(N)(w)$ is a localization of N). If $w_1 \subseteq w_2$ we have the obvious map

$$D^v(N)(w_1) = \mathfrak{R}(v \cup w_1) \otimes_{\mathfrak{R}(v)} N \rightarrow \mathfrak{R}(v \cup w_2) \otimes_{\mathfrak{R}(v)} N$$

given by $\mathfrak{R}(v \cup w_1) \rightarrow \mathfrak{R}(v \cup w_2)$. The quasi-coherence of $D^v(N)$ follows from the definition of $D^v(N)$. Given a quasi-coherent \mathfrak{R} -module M we have

$$\text{Hom}_{\mathfrak{Qco}(X)}(M, D^v(N)) \rightarrow \text{Hom}(H^v(M), N).$$

On the other hand, given $H^v(M) = M(v) \rightarrow N$, then for any w we get

$$\mathfrak{R}(v \cup w) \otimes_{\mathfrak{R}(v)} M(v) \rightarrow \mathfrak{R}(v \cup w) \otimes_{\mathfrak{R}(v)} N = D^v(N)(w).$$

But we have $M(w) \rightarrow M(v \cup w)$, so composing we get $M(w) \rightarrow D^v(N)(w)$ with the required compatibility. Then it is not hard to check that we have the required adjoint functor. \square

Example. We consider $\Omega\text{co}(\mathbf{P}^2(A))$. Then, if N is an $\mathfrak{R}(\{0, 1\})$ -module, the quasi-coherent \mathfrak{R} -module $D^{\{0,1\}}(N)$ is given by

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \nearrow & \uparrow & \nwarrow & \\
 N & & 0 & & 0 \\
 \nearrow & \nwarrow & \nearrow & \nwarrow & \nearrow \\
 N & & N & & 0
 \end{array}$$

The next result is also standard in algebraic geometry (cf. the comment preceding Proposition 3.1).

Corollary 3.2. *With the previous notation, we have:*

- *If N is injective then $D^v(N)$ is an injective \mathfrak{R} -module.*
- *If $N' \rightarrow N \rightarrow N''$ is an exact sequence of $\mathfrak{R}(v)$ -modules then $D^v(N') \rightarrow D^v(N) \rightarrow D^v(N'')$ is also exact.*

In particular an injective resolution

$$0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$$

gives an injective resolution

$$0 \rightarrow D^v(N) \rightarrow D^v(E^0) \rightarrow D^v(E^1) \rightarrow \dots$$

Proof. Immediate. \square

From the previous Corollary we have

Corollary 3.3. *Given N and v as above*

$$\text{Ext}_{\Omega\text{co}(X)}^i(M, D^v(N)) \cong \text{Ext}_{\mathfrak{R}(v)}^i(H^v(M), N).$$

The next Corollary says that if M has finite projective dimension then “locally M has finite projective dimension”.

Corollary 3.4. *If $\text{projdim } M < \infty$ then $\text{projdim } M(v) < \infty$ for every v .*

Now we focus our attention in proving the converse of Corollary 3.4. To do this we will use a special case of the $N, v, D^v(N)$ construction above. Given an \mathfrak{R} -module $M, M \neq 0$, choose $v \in Q$ maximal (with respect to \subseteq) so that $M(v) \neq 0$. Then if $v \subsetneq w$ we have $\mathfrak{R}(w) \otimes_{\mathfrak{R}(v)} M(v) = 0$. Let $N = M(v)$ and we use this v to construct $D^v(N)$. Now also note that starting with M and $\mathfrak{R}(v)N$, any $M(v) \rightarrow N$ gives $M \rightarrow D^v(N)$. We use this procedure in the special case above where we chose v maximal such that $M(v) \neq 0$ and let $N = M(v)$. So letting

$M(v) \rightarrow N = M(v)$ be id we get a morphism $M \rightarrow D^v(N)$ and so an exact sequence

$$0 \rightarrow K_v \rightarrow M \rightarrow D^v(N) \rightarrow C_v \rightarrow 0.$$

Moreover, if the previous maximal v is not unique and we denote by B the set of maximal elements we have the exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow \oplus_{v \in B} D^v(N) \rightarrow C \rightarrow 0.$$

Definition 3.5. By $Supp(M)$ for any quasi-coherent \mathfrak{R} -module M we mean the set of $v \in Q$ such that $M(v) \neq 0$.

By the construction of $D^v(N)$ from M as above it is easy to see that $Supp(D^v(N)) \subseteq Supp(M)$, so also $Supp(K), Supp(C) \subseteq Supp(M)$. But also we see (by the construction) that $v \notin Supp(K), Supp(C)$ since $(M(v) \rightarrow D^v(N)(v)) = id_N$ for such v . So $Supp(K), Supp(C) \subsetneq Supp(M)$ when $M \neq 0$.

Remark. It follows from the previous definition that $Supp(M) = \emptyset$ if, and only if, $M = 0$. If $|Supp(M)| = 1$ then $Supp(M)$ is of the form $\{i\}$ for some i , $0 \leq i \leq n$. If $w \in Supp(M)$ and $w' \subseteq w$ then $w' \in Supp(M)$. If $|Supp(M)| = 1$ and $Supp(M) = \{i\}$ then choosing v as above (maximal such that $M(v) \neq 0$) we see that $v = \{i\}$ and that with $N = M(\{i\})$ we have in fact $M = D^v(N)$.

Lemma 3.6. For any M , $projdim M < \infty$ if, and only if, $Ext^i(M, D^v(N)) = 0$ for $i \gg 0$ and for any N, v .

Proof. The condition is clearly necessary. Now let us assume the condition. We want to prove $Ext^i(M, U) = 0$ for $i \gg 0$ and any quasi-coherent module U . If $Supp(U) = \emptyset$ there is nothing to prove. If $|Supp(U)| = 1$, then $U = D^v(N)$ for some N, v and so we have $Ext^i(M, U) = 0$ for $i \gg 0$ by hypothesis. So we proceed by induction on $|Supp(U)|$. But given $U \neq 0$ we construct

$$0 \rightarrow K \rightarrow U \rightarrow \oplus_{v \in B} D^v(U(v)) \rightarrow C \rightarrow 0$$

as above (letting B be the set of maximal elements with $U(v) \neq 0$, for all $v \in B$). Then since $Supp(K), Supp(C) \subsetneq Supp(U)$ we use our induction hypothesis and easily get $Ext^i(M, U) = 0$ for $i \gg 0$. \square

Corollary 3.7. Let M be a quasi-coherent \mathfrak{R} -module. Then $projdim M < \infty$ if, and only if, $projdim M(v) < \infty$ for all $v \in Q$. Moreover, if $projdim M(v) < s$ for all $v \in Q$, then $projdim M < s + n$.

Proof. The first statement follows by the isomorphism

$$Ext^i(M, D^v(N)) \cong Ext^i(M(v), N).$$

To see the second one, let N be any quasi-coherent \mathfrak{R} -module and let w_N be the following non negative integer: $w_N = \max\{j \mid Supp(N) \text{ contains a subset of cardinality } j\}$.

We consider the exact sequence given in the proof of the Lemma 3.6:

$$0 \rightarrow K \rightarrow N \rightarrow \oplus_{v \in B} D^v(N) \rightarrow C \rightarrow 0,$$

which splits into two short exact sequences:

$$0 \rightarrow K \rightarrow N \rightarrow L \rightarrow 0,$$

$$0 \rightarrow L \rightarrow \oplus_{v \in B} D_v(N) \rightarrow C \rightarrow 0.$$

From the second short exact sequence we get the long exact sequence:

$$\cdots \rightarrow \text{Ext}^{i-1}(M, C) \rightarrow \text{Ext}^i(M, L) \rightarrow \text{Ext}^i(M, \oplus_{v \in B} D_v(N)) \rightarrow \cdots.$$

Then, we know, by hypothesis, that

$$\text{Ext}^i(M, \oplus_{v \in B} D_v(N)) = \oplus_{v \in B} \text{Ext}^i(M(v), N(v)) = 0$$

for all $i > s$. Hence if $w_N = 1$ we have $\text{Ext}^i(M, N) = 0$ for all $i > s$ because N is a direct sum of $D^v(T)$ for several objects T and vertices v . Then we can prove by induction on w_N that $\text{Ext}^i(M, N) = 0$ for all $i > s + w_N - 1$. If $w_N = 1$ the result is proved above. So let N such that $w_N = t$. Then, by the construction of the exact sequence above, we deduce that $w_C < t$ and $w_K < t$. Therefore, by induction, $\text{Ext}^{i-1}(M, C) = 0$ for all $i - 1 > s + w_C - 1$. This implies that $\text{Ext}^i(M, L) = 0$ for all $i > s + w_L - 1$ (because $w_L > w_C$). So, from the first short exact sequence, we get the long exact sequence of Ext : $\cdots \rightarrow \text{Ext}^i(M, K) \rightarrow \text{Ext}^i(M, N) \rightarrow \text{Ext}^i(M, L) \rightarrow \cdots$ and, again by induction applied to K , we conclude that $\text{Ext}^i(M, N) = 0$ for all $i > s + w_N - 1$ (note that $w_L = w_N$).

If we take N such that $w_N = n + 1$, we immediately get that $\text{projdim } M \leq s + n$. \square

Note. We also know $\text{injdim } M < \infty$ if, and only if, $\text{injdim } M(v) < \infty$ for all v . In fact $\text{injdim } M = \sup_v \text{injdim } M(v)$. As a result of the previous Corollary, the corresponding statement for $\text{projdim } M$ is not true.

Now we shall find a family of generators for $\Omega\text{co}(X)$ with finite projective dimension. We have the family of $\mathcal{O}(k)$, $k \in \mathbb{Z}$ for $\mathbf{P}^n(A)$. These give the family $\{i^*(\mathcal{O}(k)) : k \in \mathbb{Z}\}$, where $i : X \hookrightarrow \mathbf{P}^n(A)$ (see [21, pg. 120] for notation and terminology) we will let $\mathcal{O}(k)$ denote $i^*(\mathcal{O}(k))$. Since $\text{projdim}_{\mathfrak{K}(v)} \mathcal{O}(k)(v) = 0$ for all $k \in \mathbb{Z}$ and all vertex v , we get, by Corollary 3.7, that $\text{projdim } \mathcal{O}(k) \leq n$ for all $k \in \mathbb{Z}$ (so for an example, by [21, Theorem 5.1(c)], $\text{Ext}^n(\mathcal{O}(0), \mathcal{O}(-n-1)) \neq 0$ in the $X = \mathbf{P}^n(A)$ case, so we get $\text{projdim } \mathcal{O}(0) = n$).

It was proved by Serre (see for example [21]) that every coherent sheaf on $X \subseteq \mathbf{P}^n(A)$ is the quotient of a finite sum of elements of the family $\{\mathcal{O}(k) : k \in \mathbb{Z}\}$. But every quasi-coherent sheaf on X is the filtered union of coherent subsheafs. So we get that $L = \sqcup_{k \in \mathbb{Z}} \mathcal{O}(k)$ is a generator for $\Omega\text{co}(X)$. Furthermore we know that $\text{projdim } L \leq n < \infty$. (For a different, and more general, way to get this family of generators with finite projective dimension see [24, Proposition 2.3]).

Now recall that for a Gorenstein ring B (here commutative noetherian and $\text{injdim } B < \infty$) we have

$$\text{projdim } L < \infty \Leftrightarrow \text{injdim } L < \infty$$

for any B -module L (see [29]). Now suppose that $X \subseteq \mathbf{P}^n(A)$ is such that $\mathfrak{R}(v)$ is Gorenstein, for any vertex v (X will be called, as usual, a locally Gorenstein scheme). Then putting all of the above together we get.

Theorem 3.8. *If $\mathcal{A} = \mathfrak{Qco}(X)$ for $X \subseteq \mathbf{P}^n(A)$ a locally Gorenstein scheme, then \mathcal{A} is a Gorenstein category.*

Lemma 3.9. *Let M be a quasi-coherent \mathfrak{R} -module, and let*

$$0 \rightarrow M \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \cdots$$

be an injective resolution of M . Then

$$0 \rightarrow M(v) \rightarrow H^v(\mathcal{E}_0) \rightarrow H^v(\mathcal{E}_1) \rightarrow H^v(\mathcal{E}_2) \rightarrow \cdots$$

is an injective resolution of $M(v)$.

Proof. It is immediate, because the functor $H^v(-)$ is exact and preserves injective objects. \square

As a consequence we get that for a quasi-coherent sheaf being Gorenstein injective is a local property.

Corollary 3.10. *Let $X \subseteq \mathbf{P}^n(A)$ be a locally Gorenstein scheme and M be a quasi-coherent \mathfrak{R} -module over X . Then M is Gorenstein injective if and only if $M(v)$ is a Gorenstein injective $\mathfrak{R}(v)$ -module, for all vertex v .*

Proof. We shall use the pair of adjoint functors (H^v, D^v) (with $v \in Q$) obtained in Proposition 3.1.

\Rightarrow) Let

$$\cdots \rightarrow \mathcal{E}_{-2} \rightarrow \mathcal{E}_{-1} \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \cdots$$

be an exact sequence of injective quasi-coherent \mathfrak{R} -modules such that $M = \ker(\mathcal{E}_0 \rightarrow \mathcal{E}_1)$. Then, for a fixed vertex v , we have an exact sequence of injective $\mathfrak{R}(v)$ -modules

$$\cdots \rightarrow H^v(\mathcal{E}_{-2}) \rightarrow H^v(\mathcal{E}_{-1}) \rightarrow H^v(\mathcal{E}_0) \rightarrow H^v(\mathcal{E}_1) \rightarrow H^v(\mathcal{E}_2) \rightarrow \cdots$$

Then if we take an integer sufficiently large in absolute value, and apply Lemma 3.9 and [17, Theorem 9.1.11(7)] we have that $H^v(\mathcal{E}_{-m}) \rightarrow H^v(\mathcal{E}_{-m+1}) \rightarrow \cdots \rightarrow H^v(\mathcal{E}_1) \rightarrow H^v(\mathcal{E}_0) \rightarrow \cdots$ remains exact when we apply the functor $\text{Hom}(E, -)$, for all $m \geq 0$ and for all injective $\mathfrak{R}(v)$ -module E (in fact the previous is a left $\mathcal{I}nj$ -resolution, see page 167 of [17]). So $M(v)$ is Gorenstein injective.

\Leftarrow) Let M be a quasi-coherent \mathfrak{R} -module such that $M(v)$ is Gorenstein injective $\mathfrak{R}(v)$ -module. Since $\mathfrak{Qco}(X)$ is a Gorenstein category we may find an exact sequence $0 \rightarrow M \rightarrow G \rightarrow L \rightarrow 0$ with G Gorenstein injective and $L \in \mathcal{L}$. Since $M(v)$ and $G(v)$ are Gorenstein injective (the last by the previous implication) it follows that $L(v)$ is also Gorenstein injective (\mathcal{L}^\perp is a coresolving class). Then $L(v)$ is Gorenstein injective with finite projective dimension, hence an injective module, for all v . Now we take the injective cover of G (which is an epimorphism with a Gorenstein injective kernel) so we get the exact sequence $0 \rightarrow K \rightarrow E \rightarrow G \rightarrow 0$

with E injective and K Gorenstein injective. Now we make the pull-back square of $E \rightarrow G$ and $M \rightarrow G$,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K & \xlongequal{\quad} & K & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & L \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Then U is a quasi-coherent module with finite projective dimension, because it is part of the exact sequence $0 \rightarrow U \rightarrow E \rightarrow L \rightarrow 0$, and the class \mathcal{L} is a resolving class. Furthermore $U(v)$ is a Gorenstein injective $\mathfrak{R}(v)$ -module, for all v , because it is in the middle of the exact sequence $0 \rightarrow K \rightarrow U \rightarrow M \rightarrow 0$. It follows that $U(v)$ is an injective $\mathfrak{R}(v)$ -module, for all v , so U is an injective quasi-coherent \mathfrak{R} -module. So again, since \mathcal{L}^\perp is coresolving, we conclude that M is a Gorenstein injective quasi-coherent \mathfrak{R} -module. \square

Remark. It is easy to see that the methods of this section apply to other schemes. One of the main properties we require of such a scheme is that the associated quiver Q can be chosen so that (with the obvious notation) each $\mathfrak{R}(v) \rightarrow \mathfrak{R}(w)$ is a localization. This is the case, for example, of toric varieties. We also point out that the category of quasi-coherent sheaves over these schemes is equivalent to a quotient category $S - gr/\mathcal{T}$, for a suitable graded ring S and torsion class \mathcal{T} (see [8]). This fact could be useful in order to give a new focus in the topic treated in this paper, taking into account that the Gorenstein property in graded rings is well-behaved (see [3, 4]).

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